The Identity Crisis

Teacher: *Anything multiplied by one is equal to itself.*

Student: *Duh!*

As with many algorithms, the underlying mathematical structure that makes the algorithm work is usually hidden from the learner of mathematics. It is important that teachers of mathematics learn about the hidden mathematical structures and use these in making pedagogical and curricular decisions. Liping Ma (1999) identified this as an aspect of profound understanding of fundamental mathematics. According to the Common Core State Standards for Mathematics (2010), "mathematically proficient students look closely to discern a pattern or structure" (p.8).

In this article I attempt to connect the concept of additive and multiplicative identity and identity properties with several mathematical situations that relate to equivalence. I hope to show how this ties together arithmetic and algebra. It is not the intention to provide a detailed curricular outline, but rather an idea for a possible curricular thread that provides coherence, especially to middle school mathematics—the vital bridge between elementary and secondary mathematics.

In this article, I will focus more on the multiplicative identity and related properties. The concepts of additive and multiplicative inverse will also be important in this discussion. While the latter concepts appear to be more difficult for learners of mathematics, often the identity properties are as seen as trivial and thus not important. They are neither, as I hope to show here. They provide fundamental mathematical structures that make much of the mathematics that deals with equivalence understandable as well as connectable.

I will consider the following properties:

Additive identity: $n+0=n$, $(n \in \Re)$
Additive inverse: $n + {}^{-}n = 0$, $(n \in \Re)$
Multiplicative identity: $n \times 1 = n$, $(n \in \Re)$
Multiplicative inverse: $n \times \frac{1}{n} = 1$, $(n \in \Re / \{0\})$

Early on students need to focus on the various ways to represent 1 and 0. For 1, the multiplicative identity, I will make use of unity fractions, i.e. fractions with a value 1. I will present several ideas with numbers in a developmental order (equivalence in place value, measurement, fractions, and proportions) and then connect this with algebra. I will conclude with some final remarks.

<u>Place value: renaming in another basic unit</u>

For students to become fluent with numbers in the decimal number system it is important that they can *rename* a number by choosing another basic unit. This is related to determining the *equivalence* between a quantity named in one basic unit and named in another basic unit. This requires some preparation by learning to represent unity fractions, which have numerators and denominators that have the same value, but are in different units. For example, when students learn that 1 tens has the same value as 10 ones, they can express this relationship in a unity fraction with value 1 as follows:

$\frac{1 tens}{10 ones} = 1$, but also: $\frac{10 ones}{1 tens} = 1$.

As a scaffold to this idea students might first investigate equivalences with money, such as 1 dollar and 10 dimes, and 1 dime and 10 cents. Students first learn these equivalences with concrete materials, such as base-10 blocks. But for students to become flexible thinkers in mathematics they need to learn to rename numbers in a different unit. For example, they need to be able to think of 340 as 34 tens when dividing this quantity by 4,

using the long division algorithm. Moving them toward the underlying structure of equivalence in place value as I show here, has advantages for the examples I show in the remainder of the article.

If we wish to rename in a higher unit we would use the first unity fraction and if we wish to rename in a lower unit we would use the second unity faction.

Equivalence is established using the multiplicative identity, namely the product of any quantity and 1 is equal to that quantity.

Let's begin with renaming 50 ones in a higher unit (tens):

$$50ones = 50ones \times \frac{1tens}{10ones} = \frac{50ones}{10ones} \times 1tens = 5tens$$

Next we rename 50 ones in a lower unit (tenths):

$$50ones = 50ones \times \frac{10tenths}{1ones} = \frac{50ones}{1ones} \times 10tenths = 500tenths$$

<u>Converting one unit of measure in another</u>

Students frequently do not remember whether they should multiply or divide when converting one unit of measurement to another, say inches to feet or feet to inches. Doing conversions in the metric system is very similar to renaming a number in a higher or lower basic unit. We need to design unity fractions that express the conversion ratio, for example:

$$\frac{1m}{100cm} = 1, \text{ and } \frac{100cm}{1m} = 1$$

To convert, for example, a centimeter measure to meters, one would proceed to use the unity ratio that has the meter measure in the numerator as follows:

$$125cm = 125cm \times \frac{1m}{100cm} = \frac{125cm}{100cm} \times 1m = 1.25m$$

Doing conversions in the Customary system works in exactly the same fashion:

$$48inches = 48inches \times \frac{1foot}{12inches} = \frac{48inches}{12inches} \times 1foot = 4feet$$

And, finally conversion between measures in both systems:

$$3inches = 3inches \times \frac{2.54cm}{1inches} = \frac{3inches}{1inches} \times 2.54cm = 7.62cm$$

<u>Equivalent fractions and simplifying fractions</u>

Frequently students are taught that they must do to the bottom number what they do to the top number when finding equivalent fractions or when simplifying fractions to lowest terms (sometimes called “reducing,” which can be a very confusing term for students). This “rule” is vague and does not appear to have a mathematical foundation. It can lead to erroneous interpretations, such as: $\frac{2}{3} = \frac{5}{6}$. In this case three was added to both the numerator and the denominator.

We must help students see that addition does not work here, but rather that multiplication does. This is a pedagogical dilemma, because we generally don’t teach multiplication with fractions until after addition and subtraction with fractions. This is possibly why teachers sometimes resort to stating the above rule.

In many cases of equivalence we can appeal to the identity properties and so we can here. Let me first show an example of how the multiplicative identity property can be employed effectively for finding equivalent fractions. I will then briefly show how this can be scaffolded with diagrams. Let’s look at finding some equivalent fractions for $\frac{12}{15}$

$$\frac{12}{15} = \frac{12}{15} \times 1 = \frac{12}{15} \times \frac{2}{2} = \frac{24}{30}$$

and

$$\frac{12}{15}=\frac{12}{15}\times 1=\frac{12}{15}\times\frac{\frac{1}{3}}{\frac{1}{3}}=\frac{4}{5}$$

This last example can be seen as a simplification (i.e. renaming the fraction in lower or lowest terms). We can arrive at that result in a different manner that prepares students for simplifying algebraic fractions. Often this strategy is called (prime) factor and cancel. The term cancel is vague and leads to misconceptions, for example as in:

$\frac{x^2}{x+1}=x$, where students factor x^2 as $x\cdot x$ and "cancel" one of these against the x in the denominator, which is not valid. I will return to algebraic representations later.

First let's consider how using the identity property works for numerical fractions. For the factoring step, I do not encourage students to do a complete (prime) factorization of the numerator and denominator first. Rather, I encourage them to see if they can find any common factor, and not necessarily the largest one right away. Imagine doing prime factorization for $\frac{120}{150}$, a fairly cumbersome task, while with number sense it can be obvious to continue working with $\frac{12}{15}$, as we do here.

$$\frac{12}{15}=\frac{3\times 4}{3\times 5}=\frac{3}{3}\times\frac{4}{5}=1\times\frac{4}{5}=\frac{4}{5}.$$

Note that it is advantageous to keep common factors vertically aligned, so that we can isolate the unity fraction(s) more easily. It is also clear that we do not resort to crossing out the common factors (cancelling). Instead, I often draw a big numeral one around the unity fraction, such as shown in figure 1. INSERT FIGURE 1 HERE

When the fraction is not in lowest terms we can encourage the students to continue with the process until there are no common factors. As a next step we can encourage writing a

fraction in lowest terms in the most efficient way. For example:

$$\frac{120}{150}=\frac{30\times 4}{30\times 5}=\frac{30}{30}\times\frac{4}{5}=1\times\frac{4}{5}=\frac{4}{5}$$

I now want to return to a diagrammatic scaffold to show that renaming a fraction (finding equivalent fractions) is related to multiplying by a unity fraction. As an example I will rename one half as three sixths (see Figure 2).

INSERT FIGURE 2 HERE

In this way, through multiple examples, students can abstract that repartitioning a fractional part is numerically represented by a multiplication by a unity fraction, and this, in turn, represents the identity property. Note that I repartition the fraction perpendicularly. In this way I create an array structure, which inherently is multiplicative in nature. As a step before a diagrammatic representation we can use concrete materials, such a paper folding.

Being able to determine equivalent fractions is a necessary skill in preparation for adding and subtracting fractions. By example, I will demonstrate how the identity property can help with this process. In the beginning I do not encourage students to add and subtract using the lowest common denominator. If the sum or difference is not in lowest terms we can ask students to simplify these afterward. Let’s look at the sum of one half and one third:

$$\frac{1}{2}+\frac{1}{3}=\frac{1}{2}\times 1+\frac{1}{3}\times 1$$

We now must find unity fractions so that each of the addends is renamed to the same denominator as follows:

$$\frac{1}{2}+\frac{1}{3}=$$
$$\frac{1}{2}\times 1+\frac{1}{3}\times 1=$$
$$\frac{1}{2}\times\frac{3}{3}+\frac{1}{3}\times\frac{2}{2}=$$
$$\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

In general:

$$\frac{p}{a}\pm\frac{q}{b}=$$
$$\frac{p}{a}\times\frac{b}{b}+\frac{q}{b}\times\frac{a}{a}=$$
$$\frac{pb\pm qa}{ab}$$

Cross-multiplying in proportions

Labato and Ellis (2010), make a very compelling case for de-emphasizing the use of cross multiplication with proportions. They claim that this does not lead to an understanding of proportion. They explain that, "performing cross multiplication involves momentary suspension of sense making" (p.44). For a detailed discussion please consult this excellent source.

Because the use of cross multiplication for solving proportions is so prevalent in classrooms and textbooks, I do think it is important to know why this short-cut works. In that case the question is, why are the cross products of two equivalent ratios equal? The same argument presented below holds for two equivalent fractions. Once again, the mathematical explanation lies in the multiplicative identity property: $n\times 1=n$. We also

need to understand how to represent 1 with unity ratios: $1 = \frac{p}{p},\ p \neq 0$. This way we can look at the identity property as $n \times \frac{p}{p} = n$

Next, I present a generalized example of a proportion (an equivalence of two ratios). Note how I have multiplied both ratios by 1. This maintains the equivalence by the identity property. Note how I have chosen a unity ratio on both sides in a similar fashion as I chose unity fractions for the addition and subtraction of fractions.

Numerical example:	Generalized:
$\frac{4}{5} = \frac{12}{15} \Rightarrow$	$\frac{a}{b} = \frac{c}{d} \Rightarrow$
$\frac{4}{5} \times \frac{15}{15} = \frac{12}{15} \times \frac{5}{5} \Rightarrow$	$\frac{a}{b} \times \frac{d}{d} = \frac{c}{d} \times \frac{b}{b} \Rightarrow$
$\frac{4 \times 15}{5 \times 15} = \frac{12 \times 5}{15 \times 5} \Rightarrow 4 \times 15 = 12 \times 5$	$\frac{a \times d}{b \times d} = \frac{c \times b}{d \times b} \Rightarrow a \times d = c \times b$

Since the denominators are equal or common (commutative property!) in the third step, we may conclude that the numerators must be equal (by equivalence). The numerators are the "cross products." This helps us see that using cross products is a "short-cut." It leaves out the fact that the identity property, used with unity ratios, provides this result. This is the problem in many rules and algorithms, namely the mathematical structure is hidden. My opinion is that we need to teach the structure explicitly until it is internalized and the students can use such a short-cut with understanding.

Below is an alternative to the above derivation, where we compare denominators instead of numerators:

Numerical Example: Generalized

$$\frac{4}{5}=\frac{12}{15}\Rightarrow$$
$$\frac{4}{5}\times\frac{12}{12}=\frac{12}{15}\times\frac{4}{4}\Rightarrow$$
$$\frac{4\times 12}{5\times 12}=\frac{12\times 4}{15\times 4}\Rightarrow 5\times 12=15\times 4$$

$$\frac{a}{b}=\frac{c}{d}\Rightarrow$$
$$\frac{a}{b}\times\frac{c}{c}=\frac{c}{d}\times\frac{a}{a}\Rightarrow$$
$$\frac{a\times c}{b\times c}=\frac{c\times a}{d\times a}\Rightarrow b\times c=d\times a$$

What we can see here is that the numerators are equal (commutative property). So by equivalence the denominators are equal. Therefore, the cross products represent equal denominators when we have created common numerators.

A third way to look at this is by multiplying both sides by the reciprocal of one of the ratios (fractions) to create a unity fraction, thus creating a denominator and numerator that are equal.

Numerical Example:

$$\frac{4}{5}=\frac{12}{15}\Rightarrow$$
$$\frac{4}{5}\times\frac{15}{12}=\frac{12}{15}\times\frac{15}{12}\Rightarrow$$
$$\frac{4\times 15}{5\times 12}=1\Rightarrow$$
$$4\times 15=5\times 12$$

Generalized:

$$\frac{a}{b}=\frac{c}{d}\Rightarrow$$
$$\frac{a}{b}\times\frac{d}{c}=\frac{c}{d}\times\frac{d}{c}\Rightarrow$$
$$\frac{a\times d}{b\times c}=1\Rightarrow$$
$$a\times d=b\times c$$

Here we make good use that the product of a number and its reciprocal is the multiplicative identity, i.e. 1.

Connections to algebra: Algebraic fractions and solving linear equations

The identity properties work in the same way with algebraic fractions as demonstrated earlier with numerical fractions. Students need to learn to construct algebraic unity fractions, be cognizant of the domain restrictions, and they need to have good facility with algebraic multiplication and factoring.

Below I show the use of the identity properties to

a) make equivalent algebraic fractions,

b) simplify algebraic fractions, and

c) subtract (or add) algebraic fractions.

Note how finding equivalent fractions with common denominators and simplifying fractions (if needed for expressing sum or difference in lowest terms) are needed for the addition and subtraction of algebraic fractions. In the examples below I use the multiplication symbol that is more common in elementary mathematics. This is particularly helpful for demonstrating the scalar multiplication by 1. I also want to maintain that connection between arithmetic and algebra for the purpose of this article.

a)

$$\frac{x+1}{x-1} = \frac{x+1}{x-1} \times 1 = \frac{x+1}{x-1} \times \frac{x+1}{x+1} = \frac{(x+1)(x+1)}{(x-1)(x+1)} = \frac{x^2+2x+1}{x^2-1}$$

b)

$$\frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = \frac{x+1}{x+1} \times (x-1) = 1(x-1) = x-1$$

c)

$$\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{x+1}{x-1} \times 1 - \frac{x-1}{x+1} \times 1 =$$
$$\frac{x+1}{x-1} \times \frac{x+1}{x+1} - \frac{x-1}{x+1} \times \frac{x-1}{x-1} =$$
$$\frac{x^2+2x+1}{x^2-1} - \frac{x^2-2x+1}{x^2-1} = \frac{4x}{x^2-1}$$

For solving linear equations we need to employ the additive and multiplicative identity property and the additive and multiplicative inverses. Let us consider a two-step equation $3x+7=16$. Traditionally students would employ a vertical method and subtract

seven from both sides. Then students divide both sides by three. It is well documented that students have difficulty with solving two-step linear equations. I recommend limiting the method to adding the additive inverse (opposite) to both sides and multiplying both sides with the multiplicative inverse. I also encourage maintaining a horizontal format, just as we have done in all cases before. Last, the order in which these two steps happen is not important, as I will demonstrate here. The figure on the right may seem more complex, but it will pay off in the case of fractional coefficients. Try to solve $\frac{1}{3}x + \frac{2}{3} = 2$ in both ways and you may find that first multiplying both sides by the reciprocal of 1/3 is quite advantageous.

$$\begin{aligned}
&3x + 7 = 16 \Rightarrow \\
&3x + 7 + {}^{-}7 = 16 + {}^{-}7 \Rightarrow \\
&3x + (7 + {}^{-}7) = 16 + {}^{-}7 \Rightarrow \\
&3x + (0) = 9 \Rightarrow \\
&3x = 9 \Rightarrow \\
&(\frac{1}{3})3x = (\frac{1}{3})9 \Rightarrow \\
&(\frac{3}{3})x = 3 \Rightarrow \\
&x = 3
\end{aligned}$$

$$\begin{aligned}
&3x + 7 = 16 \Rightarrow \\
&(\tfrac{1}{3})(3x + 7) = (\tfrac{1}{3})16 \Rightarrow \\
&(\tfrac{1}{3})3x + (\tfrac{1}{3})7 = \frac{16}{3} \Rightarrow \\
&\frac{3}{3}x + \frac{7}{3} = \frac{16}{3} \Rightarrow \\
&x + \frac{7}{3} = \frac{16}{3} \Rightarrow \\
&x + \frac{7}{3} + ({}^{-}\frac{7}{3}) = \frac{16}{3} + ({}^{-}\frac{7}{3}) \Rightarrow \\
&x + (\frac{7}{3} + ({}^{-}\frac{7}{3})) = \frac{9}{3} \Rightarrow \\
&x = 3
\end{aligned}$$

Final Remarks

In this article, I have not exhausted all instances where the identity properties are related to equivalence. For example, it can be demonstrated, using the multiplicative identity property, that dividing a quantity by a fraction is equivalent to multiplying by its reciprocal. I have attempted to demonstrate that a focus on mathematical properties is a fruitful endeavor. This helps us reveal a structure that allows us and our students to see

many pieces of the curriculum as connected. The gift of focusing on mathematical properties is that these function as mathematical anchors throughout the K-12 curriculum, thus building vertical coherence. Your students will no longer feel that most things they learn in math have nothing to do with each other and need to be learned as separate. I do not object to students using short cuts, but these need to be a result of insight, not an unquestioned and silent acceptance of quasi-mathematical rules.

References

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Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates.

The Council of Chief State School Officers and the National Governors Association Center for Best Practices (2010). *Common core state standards for mathematics.* Retrieved from http://www.corestandards.org/, June 2, 2010.

Figure 1

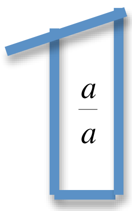
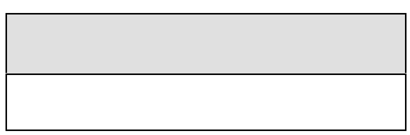
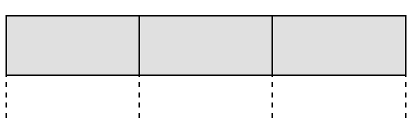


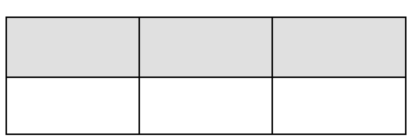
Figure 2



This figure represents $\frac{1}{2}$



Next we partition the half into three equal parts and identify all three of these parts. This way we have created $\frac{3}{3}$ of $\frac{1}{2}$



Last, we relate this partitioning of one half to the original whole. This figure represents one half as $\frac{3}{6}$ of the whole.